MATH 225 Review

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Spherical Coordinates

$x = \rho \sin\phi \cos\theta$ $y = \rho \sin\phi \sin\theta$ $z = \rho \cos\theta$
$\rho^2 = x^2 + y^2 + z^2$ $dv = \rho^2 \sin\phi \; dp \; d\phi \; d\theta$
θ is angle in xy-plane ϕ is angle from the z-axis

Change of variables

1. Substitute x and y in terms of s and t
2. Change xy region into st region
3. Use Jacobian in $\frac{\partial(x,y)}{\partial(s,t)} \; ds \; dt$

Jacobian $= \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$

Parameterized Curves

Trace out line on curve as t changes
Collision: Check if equations are equal
Cross: Change one's variables, check if collide
$\vec{v}(t) = \vec{r}\prime(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$ $\vec{v}$ is tangent to $\vec{r}(t)$'s motion
Tangent line to $\vec{r}(t)$ $l(t) = \vec{r}(t_0) + t\vec{r}\prime(t_0)$
$\vec{a}(t) = \vec{v}\prime(t) = \vec{r}\prime\prime(t) = \left\langle \frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2} \right\rangle$
Length of C: $\int_a^b ||\vec{v}|| \, dt$ where $\vec{v}$ is the velocity
Vector fields: $\vec{F}(x,y) = \langle \; , \; \rangle$
Flow line is a path along a vector field $\vec{r}\prime(t) = \vec{v} = \vec{F}(\vec{r}(t))$
Flow is the family of all flow lines
Parameterized surfaces: $\vec{r} = \langle s, t, f(s,t) \rangle$
Plane: $\vec{r}(s,t) = \vec{r_0} + s\vec{v_1} + t\vec{v_2}$

Line integrals

Circulation: If C is an oriented closed curve, the line integral of a vector field $\vec{F}$ around curve C is called the circulation of $\vec{F}$ around C
Properties:
$\int_c (\vec{F} \cdot \vec{G}) \cdot dr = \int_c \vec{F} \cdot dr + \int_c \vec{G} \cdot dr$ $\int_{c_1+c_2} \vec{F} \cdot dr = \int_{c_1} \vec{F} \cdot dr + \int_{c_2} \vec{F} \cdot dr$
$\vec{F}(x,y) = \langle x+y, y \rangle$ $\int_a^b \vec{F}(\vec{r})) \cdot \vec{r}\prime(t) dt = \int_a^b \left|\left|\vec{F}\right|\right| \, ||dt|| \cos\theta$

$$\int_C \vec{F} \cdot dr = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t)dt$$

$$\int_C \nabla f \cdot dr = f(Q) - f(P)$$

where your line goes from P to Q
Path independence $\Rightarrow$ Gradient of function and Gradient of function $\Rightarrow$ Path independence
Path independent vector fields are gradient fields
f is a potential function of $\vec{F}$
Path independent iff $\int_c \vec{F} \cdot dr = 0$ for all closed curves
curl $= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
Green's theorem: For a vector field $\vec{F}$ that has no holes on R

$$\int_c \vec{F} \cdot dr = \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA$$

Curl test: $\vec{F}$ is path independent if:

- Domain of $\vec{F}$ has no holes over region R
- If curl is zero, $\vec{F}$ is path independent
- If curl is not zero, $\vec{F}$ is not path independent

Flux integrals

$\int_s \vec{F} \cdot d\vec{A}$
For functions: $\int_R \vec{F}(x, y, f(x, y)) \cdot \langle -f_x(x, y), -f_y(x, y), 1\rangle \, dx \; dy$ $z = f(x, y)$
Parameterized surface:

$$\int \vec{F} \cdot d\vec{A} = \int_R \vec{F}\left(\vec{r}\,(s, t)\right) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) ds \; dt$$

$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ can never be zero and has to be parallel to $\vec{n}$ everywhere.
Be sure to check that $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ is parallel to $\vec{n}$
Total surface area of surface:

$$\int_S dA = \int_R \left|\left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right|\right| ds \; dt$$

Divergence

If $\vec{F} = \langle F_1, F_2, F_3\rangle$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Divergence free or solenoidal if div$\vec{F} = 0$ everywhere
Harmonic if $\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} = 0$
Divergence theorem: Where s is the given outward orientation

$$\int_s \vec{F} \cdot d\vec{A} = \int_w \text{div}\vec{F} \; dv$$

Curl of vector field in 3-space

Curl measures circulation of a vector field
If $\vec{F} = \langle F_1, F_2, F_3 \rangle$

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

If $\nabla = \langle F_1, F_2, F_3 \rangle$, $\nabla \times \vec{F} = \text{curl}\vec{F}$
Circulation of $\vec{F}$ around $\vec{n}$: $\text{circ}_{\vec{n}}\ \vec{F} = \left(\text{curl } \vec{F}\right) \cdot \vec{n}$
Curl free if $\text{curl}\vec{F} = \vec{0}$ everywhere $\vec{F}$ is defined
Stokes theorem: If S is a smooth oriented surface with piecewise smooth, oriented boundary C, and if $\vec{F}$ is a smooth vector field on an open region containing S and C, then:

$$\int_C \vec{F} \cdot dr = \int_S \text{curl } \vec{F} \cdot d\vec{A}$$

Orientation of C is determined from the orientation of S by using the right hand rule.
curl field if $\vec{F} = \text{curl } \vec{G}$ $\vec{G}$ is a vector potential of $\vec{F}$