Kyle Daling
MATH 204
Assignment 25

Section 4.5

29. a. True, $\dim V$ is the number of pivots that a matrix with column vectors given by the set given in the problem would have. There can be at least p pivots, because the vectors must be V dimensional. $\dim V$ must be less than or equal to p because the set of vectors must span a subset of V.

    b. True, if there are p V dimensional vectors in a linearly independent set, those vectors must be spanning a subset of V. Therefore there must be at least p basis vectors for V, and potentially more.

    c. True, if a set of vectors already spans V, adding another vector to the set isn't going to prevent those vectors from spanning V.

30. a. False, if V is $\mathbb{R}^3$ and the set is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\dim V = 3$ but $p = 2$.

    b. True, since every set of p vectors fails to span all of V, the set fails to be a basis for V, $\dim V > p$.

    c. False, the set of vectors doesn't have to be a basis for V, so it could be made out of vectors that are multiples of one another, which would make the set linearly dependent.

Section 4.6

12. If the null space of a 5×6 matrix is 4 dimensional, it means that that matrix must have 4 pivotless columns. If the matrix has 4 pivotless columns, then two columns must have pivots. If two columns have pivots, then the matrix must have 2 pivots, which means that the matrix has 2 row pivots. Therefore, $\dim \text{Row } A = 2$.

19. 5 linear equations in 6 variables would produce a 5×6 coefficient matrix. We are told that there the non-trivial solutions to the system span a line. Therefore, we know the system has exactly one free variable and the coefficient matrix has exactly one pivotless column. Therefore, the matrix will have 5 pivots. Since there are 5 rows and 5 pivots, there is one pivot in every row, which means the system will always be consistent for any vector in $\mathbb{R}^5$.

Section 4.7

3. Only the equation in part (ii) is satisfied. Matrix P is $\begin{bmatrix} \vec{u_1} & \vec{u_2} \end{bmatrix}_{\mathcal{W}}$. This makes P the $\mathcal{U} \to \mathcal{W}$ conversion matrix.

4. P is the $\mathcal{D} \to \mathcal{A}$ conversion matrix. Only the equation in (i) is satisfied.

6. $\mathcal{D} = \left\{ \vec{d_1} \quad \vec{d_2} \quad \vec{d_3} \right\}, \mathcal{F} = \left\{ \vec{f_1} \quad \vec{f_2} \quad \vec{f_3} \right\}, \left[\vec{f_1}\right]_{\mathcal{D}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{D}}, \left[\vec{f_2}\right]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}_{\mathcal{D}}, \left[\vec{f_3}\right]_{\mathcal{D}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}_{\mathcal{D}}$

For any $\left[\vec{x}\right]_{\mathcal{F}} \Rightarrow \left[\vec{x}\right]_{\mathcal{D}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \left[\vec{x}\right]_{\mathcal{F}} \Rightarrow \underset{\mathcal{F}\to\mathcal{D}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

b. $\left[\vec{x}\right]_{\mathcal{F}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}_{\mathcal{F}} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}_{\mathcal{D}} = \left[\vec{x}\right]_{\mathcal{D}}$

9. $\vec{b_1} = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{c_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{c_2} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

$\left[\vec{b_1}\right]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{C}} \Rightarrow x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \vec{b_1} \Rightarrow \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$

$\left[\vec{b_2}\right]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathcal{C}} \Rightarrow y_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y_2 \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \vec{b_2} \Rightarrow \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ -1 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right]$

$\therefore \left[\vec{b_1}\right]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}_{\mathcal{C}}$ and $\left[\vec{b_2}\right]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\mathcal{C}} \Rightarrow \underset{\mathcal{B}\to\mathcal{C}}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

$\therefore \underset{\mathcal{C}\to\mathcal{B}}{P} = (\underset{\mathcal{B}\to\mathcal{C}}{P})^{-1} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

10. $\vec{b_1} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{c_1} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{c_2} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\left[\vec{b_1}\right]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{C}} \Rightarrow x_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \vec{b_1} \Rightarrow \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\left[\vec{b_2}\right]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathcal{C}} \Rightarrow y_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \vec{b_2} \Rightarrow \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\therefore \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array}\right]$

$\therefore \left[\vec{b_1}\right]_{\mathcal{C}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}_{\mathcal{C}}$ and $\left[\vec{b_2}\right]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathcal{C}} \Rightarrow \underset{\mathcal{B}\to\mathcal{C}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$

$\therefore \underset{\mathcal{C}\to\mathcal{B}}{P} = (\underset{\mathcal{B}\to\mathcal{C}}{P})^{-1} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$

Chapter 4 Supplementary Problems

1. a. True, the set of all linear combinations of a set of vectors is the same as the span of those vectors which is the definition for a subspace.

b. True, if a set of vectors already spans the entirety of a space, adding another vector to that set will still allow the vectors to span the entirety of a space.

c. False, if you add a vector to a set of linearly independent vectors it is not necessarily true that the new set will be linearly independent. It is possible for the new vector to be a multiple of a vector already in the set, which would make the new set linearly dependent.

d. False, this is only one of the conditions for being a subspace. If set S spans all of V and is linearly independent then it is a basis for V.

e. True, if a set S spans all of a space, it is not necessarily true that the set is linearly independent. However, it would be possible to find a subset of S that is linearly independent and spans all of V, which would satisfy the conditions for a basis for space V.

f. True, if the set of vectors S is a basis for a space V and $\dim S =$ the number of vectors in S, then S must be a basis for V, and the vectors in S must be linearly independent.

g. False, if the plane does not contain the origin, then it can't be a subspace.

h. False, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The non pivot columns in this matrix form the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, which is a linearly independent set.

i. True, $R_1 + R_3 = R_2$ $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 + R_3 \neq R_2$. The dependency relationship between the three rows has changed by row reduction.

j. False, row operations do not change the location of pivot columns. As a result row operations do not change the dependency relations of columns. Therefore row reduction does not change the null space of a matrix.

k. False, the rank of a matrix is equal to the number of pivot columns in the matrix.

l. False, nonzero rows don't tell us anything about the null space, only the number of pivotless columns tell us anything about the null space.

m. True, if B is obtained by multiplying A by elementary matrices, then B is row equivalent to A. Therefore the two matrices will have the same pivots, and their rank will be equal.

n. False, the rows with pivots in A form the basis for Row A.

o. True, if A and B share the same rref form, then they are row equivalent. Therefore the two matrices will have the same pivots and the same bases for their row subspaces.

q. False, if rank $A = m$ there must be m pivots. If A is 2×35 there are 2 columns with pivots and 33 columns that are pivotless in A. For the transformation to be one-to-one, there must be a pivot in every column of A.

r. True, if a linear transformation is onto, there must be a pivot in every row. It there is a pivot in every row $n \geq m$. Therefore there will be m pivots, and m columns will have pivots making Rank $A = m$.

t. False, the j^{th} column of the $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ would be $\left[\vec{b_j}\right]_{\mathcal{C}}$

2. $$a\begin{bmatrix}1\\2\\-1\\3\end{bmatrix}+b\begin{bmatrix}-2\\5\\-4\\1\end{bmatrix}+c\begin{bmatrix}5\\-8\\7\\1\end{bmatrix}\Rightarrow\begin{bmatrix}1&-2&5\\2&5&-8\\-1&-4&7\\3&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\Rightarrow\begin{bmatrix}1&-2&5\\2&5&-8\\-1&-4&7\\3&1&1\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix}$$

$$\text{Basis}=\left\{\begin{bmatrix}1\\2\\-1\\3\end{bmatrix},\begin{bmatrix}-2\\5\\-4\\1\end{bmatrix}\right\}$$