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MATH 204

Assignment 16

Section 2.2

17. $AB = BC \Rightarrow ABB^{-1} = BCB^{-1} \Rightarrow A = BCB^{-1}$

18. $A = PBP^{-1} \Rightarrow AP = PBP^{-1}P \Rightarrow AP = PB \Rightarrow P^{-1}AP = P^{-1}PB \Rightarrow P^{-1}AP = B$

31. $[A|I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$
    Since A became an identity matrix I conclude that A is invertible.

32. $[A|I] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 & \dots \\ 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix}$.
    Since A didn't become an identity matrix I conclude that A is not invertible.

Section 2.3

2. The two columns are linearly dependent $\frac{-3}{2}\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

3. I can't find an easy trick that lets me know anything.
    $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \xrightarrow{rref} I_3 \therefore$ the the matrix must be invertible.

6. $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{8}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$. This matrix does not rref to an identity matrix, therefore it must not be invertible.

16. It is impossible for a 5×5 matrix to be invertible if its columns don't span all of $\mathbb{R}^5$. For a 5×5 matrix to span all of $\mathbb{R}^5$ it must have a pivot in every row, which would be 5 pivots. If the matrix doesn't span all of $\mathbb{R}^5$ then the matrix can only have at most 4 pivots. This would make it impossible for every row and column to have a pivot, which is necessary for a matrix to be an identity matrix. If a matrix is not row reducible to an identity matrix, then it is not invertible.

18. If $C\vec{x} = \vec{v}$ is consistent $\forall \vec{v} \in \mathbb{R}^6$, then C must have a pivot in each of its 6 rows, giving matrix C 6 total pivots. If C has 6 pivots it must have a pivot in every column because the matrix has exactly 6 columns. If C has a pivot in every column, and C is put into a homogeneous system $C\vec{x}_2 = \vec{0}$ only the trivial solution can exist which means there is only a unique solution for each $\vec{v} \in \mathbb{R}^6$.

19. If the columns of D, a 7×7 matrix, are linearly independent then we know that every column of D must have a pivot giving matrix D 7 total pivots. If D has 7 pivots and 7 rows then it must also be the case that D has a pivot in every row. It must also be the case that there is a unique solution for each $\vec{b} \in \mathbb{R}^7$.

20. If two square matrices have the property that $EF = I_n$, then E and F must be invertible matrices. As such they must also commute.

21. If the equation $G\vec{x} = \vec{y}$ has more than one solution for any $\vec{v} \in \mathbb{R}^n$ then matrix G must not have a pivot in every column. This would give G at most $n-1$ pivots. This would make the matrix non-invertible, making it unable to span all of $\mathbb{R}^n$.

22. If $H\vec{x} = \vec{c}$ is inconsistent for some $\vec{c}$ then there must not be a pivot in every row of H. You can't say much about $H\vec{x} = \vec{0}$ but we know that it will always be consistent and that it will only span a subspace of the dimension the column vectors of H are in.

23. If K and $n\times n$ matrix cannot be reduced to I_n then we know that the matrix must not have n pivots. This means matrix K cannot have a pivot in every column. The columns of K must then be linearly dependent.

24. If equation $L\vec{x} = \vec{0}$ has the trivial solution we don't know much about the matrix. It is possible for L to have a row without a pivot and still have the trivial solution which would make it impossible for matrix L to span all of $\mathbb{R}^n$.

Supplementary Exercises 2

1. a. True: A is $m\times n$, A^T is $n\times m$; B is $m\times n$, B^T is $n\times m$. For each case it is possible to multiply: $AB^T \to (m\times n)\times(n\times m)$, $A^TB \to (n\times m)\times(m\times n)$.

   b. False, if $AB = C$ then B must have two columns by the definition of matrix multiplication. $C = \begin{bmatrix} A\vec{B}_1 & A\vec{B}_2 \end{bmatrix}$

   c. True, A is an identity matrix and left multiplying B by A will scale the rows of B.

   d. False, matrices C and D could be scalar multiples of one another and still have $BC = BD$.

   e. False, by example $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Both A and C are non-zero matrices.

   f. False, that isn't how multiplication works. If you factored them out you would get $A(A-B)+B(A-B)$.

   g. True, an elementary matrix is an identity matrix that has had one row operation performed on it. Identity matrices have a pivots along their diagonal and zeros elsewhere. Therefore an identity matrix can only n non-zero entries. For each of the basic row operations scaling and row swap don't add another value to the elementary matrix, they just shift them around. The last operation, row interchange will add a multiple of another row to the current row. This would add another entry to the elementary matrix allowing the elementary matrix to have $n+1$ non-zero entries.

h. True, elementary matrices are square, swapping the rows and columns will give the same matrix.

i. True, elementary matrices must be square because they are identity matrices that have had one row operation done to them, and identity matrices are square.

j. False, an elementary matrix must row reduce to an identity matrix, but a square matrix doesn't have to reduce to an identity matrix. Therefore not all square matrices are products of elementary matrices.

k. True, if a 3×3 matrix has 3 pivots then it must be row reducible to an identity matrix. The sequence of elementary matrices will determine how to get the original matrix into the identity matrix.

l. False, matrix A could be 3×5 and matrix B could be 5×3 and they could be multiply to be an identity matrix, but neither A nor B are invertible because they are not square.

p. True, since we know the system has a unique solution we also know that there must be a pivot in every column of A. Therefore we know that A has 3 pivots which makes it invertible.